Shock waves, shock profiles and stability

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A quasilinear system of first-order conservation laws

$$\partial_t u + \partial_x f(u) = 0 \tag{0.1}$$

looks to be reversible in time at the first glance. Actually, it is not, because of the development of shock waves in finite time : no matter with the regularity of the initial data, discontinuities will appear at some time $T > 0$. Beyond this instant, the reversibility is lost. When the system has a physical interpretation (gas dynamics, elasticity, Maxwell's equations in a non-linear medium), the best way to understand irreversibility is to go back to a more accurate modelling of the phenonmenon under consideration. This yields to a perturbed system, where we shall consider only the most frequent terms, those of second order ;

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x (B(u^\epsilon)\partial_x u^\epsilon). \tag{0.2}$$

Hereabove, $\epsilon > 0$ is a small parameter.

Since shock waves are special cases of travelling waves of (0.1), we ask that they be associated to travelling waves of (0.2). This selection process is called the *viscosity criterion*, since the perturbation often describes the effect of a newtonian viscosity.

We describe in this course various admissibility criterion for shock waves, but focus on the viscosity criterion. Especially, we pay attention to the stability of the travelling waves (the *viscous profiles*), as it determines whether they may be observed or not.

Besides, we explain that several similarities occur between the viscous approximation of (0.1) and the approximation by conservative difference schemes. We discuss the analogous *discrete shock profiles* and their stability.

Chapter 1

Discontinous solutions of conservation laws

1.1 Notations

In this course, we consider systems (say n equations) of first-order conservation laws in one space variable $x \in I\!R$:

(1.1) $$\partial_t u + \partial_x f(u) = 0,$$

where the unknown $u(x,t)$ takes its values in an open convex set $\mathcal{U}$ of $I\!R^n$, and $f : \mathcal{U} \mapsto I\!R^n$ is a given smooth vector field. We are interested in the Cauchy problem : given an initial data $a \in L^\infty(I\!R)$, to find a solution $u : I\!R \times (0,T) \to \mathcal{U}$, where $T > 0$, which solves (1.1) and satisfies

(1.2) $$u(x,0) = a(x), \quad a.e.$$

We say that (1.1) is a **scalar equation** if $n = 1$.

A linear analysis shows that well-posedness in standard spaces (as Hölder or Sobolev spaces) is expected only if the system is **hyperbolic**. We shall make this asumption throughout this course :

> **(Hyp)** : at every point $b \in \mathcal{U}$, the jacobian matrix
> $$df(b) := \left(\frac{\partial f_i}{\partial u_j}\right)_{1 \le i,j \le n}$$
> is diagonalisable with real eigenvalues.

For technical reasons, we shall often strengthen this asumption :

> **(SHyp)** : one says that the system (1.1) is **strictly hyperbolic** at $b \in \mathcal{U}$ if $df(b)$ has n distinct real eigenvalues.

In such a case, the denote by $\lambda_1(b) < \cdots < \lambda_n(b)$ the eigenvalues. Eigenfields are denoted by $r_\alpha(b)$ ($1 \le \alpha \le n$) :

$$\sum_{j=1}^n \frac{\partial f_i}{\partial u_j} r_{\alpha,j} = \lambda_\alpha r_{\alpha,i}, \quad 1 \le i \le n.$$

The eigenspaces are denoted by $E_\alpha(b)$. Similarly, eigenforms are denoted by $l_\alpha(b)$:

$$\sum_{i=1}^n l_{\alpha,i} \frac{\partial f_i}{\partial u_j} = \lambda_\alpha l_{\alpha,j}, \quad 1 \le j \le n.$$

One always may assume that the basis are dual to each other :

$$l_\alpha \cdot r_\beta = \delta_\alpha^\beta.$$

A *characteristic field* of the system (1.1) is nothing but an eigenfield, given by an index $\alpha \leq n$. According to Lax [5], we say that the α-characteristic field is **genuinely non-linearly** (in short GNL) at a point b if the differential form $d\lambda_\alpha$ does not vanish along the eigenvector r_α : $d\lambda_\alpha(b) \cdot r_\alpha(b) \neq 0$. In such a case, we normalize $r_\alpha(b)$ and $l_\alpha(b)$ by

$$d\lambda_\alpha(b) \cdot r_\alpha(b) = 1, \quad l_\alpha(b) \cdot r_\alpha(b) = 1.$$

This choice determines uniquely r_α and l_α.

On the other hand, the α-characteristic field is said to be **linearly degenerate** (or LD) in $\mathcal{U}$ if $d\lambda_\alpha \cdot r_\alpha \equiv 0$. For such fields, there is no distinguished choice of the eigenfield.

1.2 The non-existence of classical solutions

The local-in-time well posedness of the Cauchy problem is well-known (see for instance [1]). Here is a standard result.

Theorem 1.2.1 *Let the system (1.1) be strictly hyperbolic at the point b. then there is a $\delta > 0$ such that, for any given initial data $a \in \mathcal{C}^1(I\!R)$, with values in the ball $B(b;\delta)$, the Cauchy problem admits one and only one classical solution $u \in \mathcal{C}^1(I\!R \times (0, T(a)))$. Here, the existence time $T(a) > 0$ depends on δ and $\|a'\|_\infty$.*

However, $T(a)$ is finite in general, so that this theorem is of very little help for applications. To provide an example, we need to choose a non-linear f, since a linear hyperbolic Cauchy problem is globally well-posed. The simplest example is scalar, with $f'' > 0$. Then, $v := f'(u)$ satisfies the equation $\partial_t v + v\partial_x v = 0$ whenever u is $\mathcal{C}^1$. Let $t \mapsto \gamma_y(t)$ be defined by $\gamma_y(0) = y$ and $\gamma_y'(t) = v(\gamma_y(t), t)$. Then

$$\frac{d}{dt} v(\gamma_y(t), t) = \partial_t v + \gamma_y' \partial_x v = 0,$$

so that v is constant along the curve Γ_y parametrized by γ_y, equal to $f' \circ a(y)$. Thus the slope of Γ_y is constant : $\gamma_y(t) = y + tf' \circ a(y)$. Now the equation

$$v(y + tf' \circ a(y), t) = a(y), \quad y \in I\!R$$

multiply defines v as soon as several curves Γ_y intersect. Such intersections at positive times occur provided $f' \circ a$ decreases somewhere. thus

Theorem 1.2.2 *Let $a \in \mathcal{C}^1$ be such that $d/dx(f' \circ a)(y_0) < 0$ at some point y_0. Then $T(a)$ is finite. Indeed the maximal existence time of the classical solution is*

$$T(a) = -\frac{1}{\inf_{y \in I\!R} f' \circ a(y)}.$$

For this reason, we shall always consider merely **weak solutions**, which are bounded measurable functions $u : Q_T := I\!R \times (0,T) \to \mathcal{U}$, satisfying (1.1,1.2) in the distributional sense :

$$\int_{Q_T} (u\partial_t \phi + f(u)\partial_x \phi)\, dx dt + \int_{I\!R} \phi(x,0) a(x) dx = 0, \quad \forall \phi \in \mathcal{C}^1(I\!R \times [0,T)).$$

One verifies readily that

Proposition 1.2.1 *Let Γ be a $\mathcal{C}^1$-curve within Q_T. Let u be $\mathcal{C}^1$ away from Γ, having continuous left and right limits $u^\pm$ on Γ. Then u is a weak solution of the Cauchy problem if and only if ;*

- *on the one hand, u is a classical solution away from Γ,*
- *on the other hand, along Γ, u satisfies the **Rankine-Hugoniot** condition*

$$(u^+ - u^-)\nu_t + (f(u^+) - f(u^-))\nu_x = 0, \tag{1.3}$$

$\vec{\nu}$ denoting a normal vector field to Γ.

Because f is locally lipschitz, (1.3) implies $|\nu_t| \leq M|\nu_x|$ as soon as $[u] := u^+ - u^-$ is non zero. Thus such curves of discontinuity must be parametrized in a Lipschitz way as $t \mapsto x = X(t)$ with

$$[f(u)] = \frac{dX}{dt}[u]. \tag{1.4}$$

1.3 The non-uniqueness of weak solutions

Although the introduction of weak solutions remedies to the non-existence, it is not satisfactory because it brings naturally the non-uniqueness. Again we give a scalar example. Here, $f(u) := u^2/2$ (the **Burgers equation**) and $a \equiv 0$. Then $u \equiv 0$ is a classical solution (the unique one), obviously the relevant one. But here is a one-parameter ($p > 0$) family of non-trivial weak solutions :

$$v_p(x,t) := \begin{cases} 0, & |x| > pt, \\ -2p, & -pt < x < 0, \\ 2pt, & 0 < x < pt. \end{cases}$$

To verify that v is a solution of the Cauchy problem, just apply proposition 1.2.1.

We thus have to select among weak solutions, in order to have one and only one solution to the Cauchy problem. Let us mention that this program, in its whole generality, is far to be acomplished today. Since classical solutions certainly must be admitted, we only have to select among discontinuities those which are **admissible**. To do that, we shall need to consider both stability questions and physical arguments, since systems such that (1.1) often represent some physical phenomenon.

Remarks - In the linear case, the Cauchy problem has a unique weak solution, thus admissibility criteria must become trivial for linear fluxes f. - For a scalar equation, all the reasonable criteria will be equivalent to each other. This is no longer true for systems ($n > 1$).

1.4 Lax shocks

An α-**characteristic curve** in Q_T is a lipschitz curve parametrized by a solution γ of the differential equation

$$\gamma'(t) = \lambda_\alpha(u(\gamma(t),t)).$$

It thus depends not only on the system (1.1), but indeed on the solution itself, as soon as f is not linear. Most of the proof of theorem 1.2.1 is based on a calculus along characteristics. In presence of finitely many non-intersecting discontinuities, one may proceed as follow. For the sake of simplicity, we assume that a is $\mathcal{C}^1$ away from $y = 0$, with left and right limits $a^\pm$ at zero. If $[f(a)]$ is colinear to $[a]$, say $[f(a)] = \sigma[a]$, we expect that a weak solution is smooth away a curve $\Gamma : t \mapsto (\gamma(t), t)$ originated from $(0,0)$.

Step 1. Change of variable : we define $y := x - \gamma(t)$ and $v(y,t) = u(x,t)$. Then Γ is mapped onto the axis $y = 0$. We now have two unknowns $v(y,t)$ and $\gamma(t)$. Characteristics of velocity λ_α are mapped on characteristics of velocity $\lambda_\alpha - \gamma'$.

Step 2. We proceed as in the proof of theorem 1.2.1. However, we now deal with two initial-boundary value problems, one for $y > 0$, the other one for $y < 0$. Furthermore, they couple through the Rankine-Hugoniot relation. To determine uniquely a classical solution of a first-order system in a quarter plane $y, t > 0$, one not only needs an initial data, but one needs also q independant boundary conditions, where q is the number of **incoming characteristics**, that is those of positive velocities. Thus q is the index such that

$$\lambda_{n-q}(u^+) \leq \gamma' < \lambda_{n-q+1}(u^+).$$

Similarly, to solve uniquely the problem in the left quarter $y < 0 < t$, one needs r independant boundary conditions, where

$$\lambda_r(u^-) < \gamma' \leq \lambda_{r+1}(u^-).$$

Naturally, we also need that γ' be determined.

In such an analysis, only the Rankine-Hugoniot condition can be used to determine both γ' and $q + r$ boundary conditions. Since it consists in n independent equations, this means that the boundary problem we are solving must have $q + r = n - 1$ incoming characteristics. If $q+r > n-1$, the problem would be under-determined (non-uniqueness of a piecewise-smooth solution). If $q+r < n-1$, it is over-determined (non-existence for general data).

This analysis shows that an admissible discontinuity must satisfy the inequalities

$$\lambda_{p-1}(u^-) < \gamma' \leq \lambda_p(u^-), \quad \lambda_p(u^+) \leq \gamma' < \lambda_{p+1}(u^+), \tag{1.5}$$

for some index p. We say that the discontinuity is a **p-Lax shock** if all these inequalities are strict, whereas it is called a **contact discontinuity** if $\lambda_p(u^\pm) = \gamma'$. One easily construct contact discontinuities when the p-characteristic field is LD, whereas shocks merely correspond to GNL fields.

The typical result which might be proved by the above framework is the following :

Theorem 1.4.1 *Let $a^\pm : I\!R^\pm \to \mathcal{U}$ be $\mathcal{C}^1$ functions and let $a \in L^\infty(I\!R)$ be the function whose restrictions to $I\!R^\pm$ are $a^\pm$. Let us assume the compatibility conditions :*

$\mathcal{C}^0$ compatibility

$$\exists \sigma \in I\!R, \quad [f \circ a] = \sigma[a].$$

$\mathcal{C}^1$ compatibility

$$[(df \circ a - \sigma)^2 a'] \wedge [a] = 0.$$

Let us assume that $(a^-(0), a^+(0); \sigma)$ is a p-Lax shock :

$$\lambda_{p-1}(u^-) < \sigma < \lambda_p(u^-), \quad \lambda_p(u^+) < \sigma < \lambda_{p+1}(u^+).$$

Finally, let us assume the (generic) Majda's condition *:*

$$\det(r_1(u^-), \cdots, r_{p-1}(u^-), u^+ - u^-, r_{p+1}(u^+), \cdots, r_n(u^+)) \neq 0.$$

Then the Cauchy problem admits one and only one piecewise $\mathcal{C}^1$ solution, with a discontinuity located on a $\mathcal{C}^2$ curve originated at $(x,t) = (0,0)$. locally in time.

Remarks :

- the $\mathcal{C}^k$ condition is clearly necessary for the solution to be piecewise $\mathcal{C}^k$. They together become sufficient in presence of the Lax shock condition.

- The Rankine-Hugoniot condition is symmetric in the left and right states (u^l, u^r), but the Lax shock condition is not. This causes irreversibility in the evolution.

- The Majda's criterion is the one ensuring the well-posedness of the linearized initial-boundary-value problem. Clearly, it is a generic one, in the sense that data which satisfy it are dense. This however is no longer true for systems in several space dimensions. Let us point that the criterion is always satisfied for weak shock since, as u^+ is close to u^-, $r_\alpha(u^+)$ is approximately colinear to $r_\alpha(u^-)$, whereas $u^+ - u^-$ is approximately colinear to $r_p(u^-)$. See the next section.

1.5 The Hugoniot locus

Let b be a point in $\mathcal{U}$. The **Hugoniot locus** $H(b)$ is the set of states $c \in \mathcal{U}$ such that $f(c) - f(b)$ is parallel to $c - b$. In other words, it is the projection on $\mathcal{U}$ of the sub-manifold of $\mathcal{U} \times I\!R$ defined by the Rankine-Hugoniot condition $f(c) - f(b) - \sigma(c - b) = 0$. In the linear case ($f(u) = Au$), the Hugoniot locus of b is clearly the union of lines $b + E_\alpha$, where E_α are the eigenspaces $I\!R r_\alpha$. The strictly hyperbolic case ressembles the linear one :

Theorem 1.5.1 (Lax) *Let (1.1) be strictly hyperbolic. Then the intersection of the Hugoniot locus with a small enough ball $B(b;\epsilon)$ is the union of smooth curves $H_1(b), \dots, H_n(b)$, where $H_\alpha(b)$ is tangent to $r_\alpha(b)$ at b.*

A slightly deeper analysis characterizes those weak (that is with small amplitude) discontinuities which are Lax shocks :

Theorem 1.5.2 (Lax) *Let the p-characteristic field be GNL at b and let $r_p(b)$ be normalized. Then, on $H_p(b) \cap B(b;\epsilon)$, the states c located on the side $-r_p(b)$ are right states of a Lax shock $(b, c; \sigma)$, whereas the states c located on the side $+r_p(b)$ are left states of a p-Lax shock $(c, b; \sigma)$. More precisely :*

$$\lambda_p(c) - \sigma \sim \sigma - \lambda_p(b) \sim \frac{1}{2} l_p(b) \cdot (c - b).$$

This result shows that for GNL fields, half of discontinuities satisfy the Lax criterion. In other words, given $(b, c; \sigma)$ satisfying the Rankine-Hugoniot condition, either $(b, c; \sigma)$ or $(c, b; \sigma)$ is a Lax-shock. Regarding the LD fields, one easily show that H_p is an integral curve of r_p and $\sigma = \lambda_p(b) = \lambda_p(c)$. These discontinuities are contact and must be considered as admissible.

From now on, we shall focus only on GNL fields, with possibly exceptional points where $d\lambda_p \cdot r_p$ vanish.

1.6 The vanishing viscosity method

For most of non-linear phenomenon in physics, a slightly parabolic model as

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon) = \epsilon \partial_x (B(u^\epsilon) \partial_x u^\epsilon). \tag{1.6}$$

is more accurate than the hyperbolic first-order model. Although both are strongly related to each other as $\epsilon \to 0+$, the time-irreversibility of (1.6) induces the one of (1.1). That is, the limits of solutions of (1.6) will in general be irreversible solutions of (1.1). In other words, if

u is such a limit, and if u is piecewise smooth with a discontinuity not being a contact, then $v(x,t) := u(-x,-t)$ cannot be such a limit. Although such a statement has never been proved, it fits with the experience.

Here we provide a framework which is relevant for applications in physics. It has been developped in 1987 by Kawashima in his PhD thesis. We show that it is well-suited from a mathematical point of view. Finally, we give a few examples in gas dynamics.

Let first consider a change of variable $u = f^0(w)$, where w runs through an open set $\mathcal{O}$. Such a transformation has to be designed in such a way that the system becomes as simple as possible, under a restriction that we shall point soon later. The system (1.6) is equivalent (at least for classical solutions) to

$$\partial_t f^0(w) + \partial_x f^1(w) = \epsilon \partial_x (G(w)\partial_x w), \tag{1.7}$$

where $f^1 = f \circ f^0$, $G = (B \circ f^0) df^0$.

We then suppose that the system (1.1) admits an entropy-flux pair, that is a pair of scalar functions (η, q) such that (1.1) implies $\partial_t \eta(u) + \partial_x q(u) = 0$ whenever u is a classical solution. Let us point out that this equality is no longer valid in presence of shock waves. Equivalently, (η, q) solves the linear system $dq = d\eta df$, that is

$$\frac{\partial q}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial \eta}{\partial u_j} \frac{\partial f_j}{\partial u_i}, \quad 1 \le i \le n. \tag{1.8}$$

Differentiating (1.8), one readily checks that $D^2 \eta df$ is symmetric. We also assume that η is strongly convex with respect to u :

$$D^2_{uu} \eta > 0. \tag{1.9}$$

Multiplying (1.7) to the left by $(df^0)^T D^2 \eta \circ f^0$, one obtains the following identity :

$$A^0(w)\partial_t w + A^1(w)\partial_x w - \epsilon D \partial_x^2 w = \epsilon Q(w, \partial_x w), \tag{1.10}$$

where $Q(w,\cdot)$ is quadratic, $A^j = (df^0)^T D^2 \eta (df^j)$ is symmetric and A^0 being positive definite. For instance,

$$A^0 = (df^0)^T (D^2 \eta \circ f^0) df^0.$$

Hereabove, we have $D = (df^0)^T ((D^2 \eta B) \circ f^0) df^0$.

Let us now assume that B is *symmetric and weakly-dissipative with respect to* η, that is

$$(D^2 \eta B)^T = D^2 \eta B, \quad (D^2 \eta B X | X) \ge \alpha(u) \|BX\|^2, \tag{1.11}$$

for all $X \in \mathbb{R}^n$ and a suitable positive number $\alpha(u)$. Then,

$$(DX|X) \ge \alpha(u) \|B df^0 X\|^2 \ge \beta(u) \|DX\|^2,$$

for some positive number $\beta(u)$.

At this level, one may establishes an *a priori* estimates for sufficiently smooth solutions of (1.6). Multiplying by $d\eta(u^\epsilon)$, one obtains

$$\partial_t \eta(u^\epsilon) + \partial_x q(u^\epsilon) + \epsilon (B(u^\epsilon)\partial_x u^\epsilon | D^2 \eta(u^\epsilon) \partial_x u^\epsilon) = \epsilon \partial_x (D^2 \eta(u^\epsilon) B(u^\epsilon) \partial_x u^\epsilon). \tag{1.12}$$

Let us assume for simplicity that the initial data tends to a constant $\bar{u}$ at infinity, so that we expect a solution having the same property. Up to the addition of a linear functional, one may suppose that η and $d\eta$ vanish at $\bar{u}$, so that $\eta(u)$ is positive for every $u \neq \bar{u}$. Now, integrating (1.12), one obtains formally

$$\frac{d}{dt}\int_{I\!R}\eta(u^\epsilon)dx + \epsilon\int_{I\!R}\alpha(u^\epsilon)\|\partial_x u^\epsilon\|^2 dx \leq 0. \tag{1.13}$$

For a more general data, a less simple but still good inequality is obtained but multiplying first (1.12) by a non-negative test function. However, even (1.13) is not sufficient for proving the existence of local-in-time solutions of (1.6), when B is not invertible. For instance, it gives only an L^2-estimate when $B \equiv 0$, whereas the Cauchy problem is not well-posed in L^2, except for a linear f. We last remark that the estimate is not ϵ-independant.

We now make a few structural assumptions, involving the choice of the auxiliary variables w. We first assume that both A^0 and D are block-diagonal (with same sizes $n = n_1 + n_2$) :

$$A^0 = \begin{pmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix},$$

where D_2 is invertible, that is $n_2 = \mathrm{rk}B$. We last demand that the first block of the quadratic term does not depend on the second block of w. In short, using variables $(y, z) = w$, the system (1.10) split into

$$\left\{\begin{array}{rcl} A_1^0\partial_t y + A_1^1\partial_x y & = & \epsilon Q_1(y, z; \partial_x z), \\ A_2^0\partial_t z + A_2^1\partial_x z - \epsilon D_2\partial_x^2 z & = & \epsilon Q_2(y, z; \partial_x y, \partial_x z). \end{array}\right.$$

Under these assumptions, we have

Theorem 1.6.1 *Let s be a number larger than $5/2$ and let the initial data belong to $\bar{w} + H^s(I\!R)^n$ for some constant state $\bar{w}$. Under the previous assumptions (*entropy+block-form*), there exists a $T > 0$ and a unique solution w such that*

$$\begin{array}{rcl} y - \bar{y} & \in & \mathcal{C}([0,T]; H^s) \cap \mathcal{C}^1([0,T]; H^{s-1}), \\ z - \bar{z} & \in & \mathcal{C}([0,T]; H^s) \cap \mathcal{C}^1([0,T]; H^{s-2}) \cap L^2(0,T; H^{s+1}). \end{array}$$

However, the existence time T may depend on ϵ for a fixed data.

Last, we assume that there exists a matrix $K(w)$ such that

- first, KA^0 is skew-symmetric,

- second, the symmetric part of $KA^1 + B$ is positive definite.

Then

Theorem 1.6.2 *Under the last assumption (*skew-symmetrizer*), the constant states are asymptotically stable in the following sense : if $w(t=0) - \bar{w} \in H^s(I\!R) \cap L^1(I\!R)$ (with $s > 7/2$) and if the norm of $w(t=0) - \bar{w}$ in this space is sufficiently small, then the solution w is defined for all time (that is $T = +\infty$) and moreover*

$$\|w(t) - \bar{w}\|_{H^s} = \mathcal{O}(t^{-1/4}).$$

Here, the smallness criterion depends on ϵ.

1.6.1 An example : the gas dynamics

The flow of a compressible homogneous fluid is described by its mass density ρ, its velocity field v and its specific internal energy. Assuming some Newtonian viscosity and heat conduction, the equations write, in one space variable,

$$\begin{aligned}
\partial_t\rho + \partial_x(\rho v) &= 0,\\
\partial_t(\rho v) + \partial_x(\rho v^2 + p(\rho, e)) &= \partial_x(\nu\partial_x v),\\
\partial_t\left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_x\left(\left(\frac{1}{2}\rho v^2 + \rho e + p\right) v\right) &= \partial_x(\nu v\partial_x v + \kappa\partial_x\theta).
\end{aligned}$$

Hereabove, the pressure p is a given function which satisfies a criterion ensuring the hyperbolicity of the system in the absence of dissipation. The positive coefficients ν and κ are functions of (ρ, e). Finally, $\theta(\rho, e)$ is the temperature, that is positive function such that $\theta^{-1}(de + pd(1/\rho))$ is an exact differential form dS. The function S is the "entropy" of the fluid. The hyperbolicity assumption is equivalent to say that S is a concave function of the arguments

$$u := \begin{pmatrix} \rho \\ \rho v \\ \frac{1}{2}\rho v^2 + \rho e \end{pmatrix}.$$

We choose $w := (\rho, v, \theta)^T$ as new variables. Then, with

$$A^0 = \operatorname{diag}\left(\frac{p_\rho}{\rho}, \rho, \frac{\rho e_\theta}{\theta}\right),$$

the system takes the Kawashima's form, with

$$A^1 = vA^0 + \begin{pmatrix} 0 & p_\rho & 0 \\ p_\rho & 0 & p_\theta \\ 0 & p_\theta & 0 \end{pmatrix}, \quad D = \operatorname{diag}\left(0, \nu, \frac{\kappa}{\theta}\right).$$

The skew-symmetric matrix K is

$$K = \alpha\begin{pmatrix} 0 & p_\rho & 0 \\ -p_\rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha > 0$ is chosen so that

$$\alpha < \min\left\{\frac{\nu}{\rho|p_\rho|}, \frac{4\rho\kappa}{\theta p_\theta^2}\right\}.$$

1.7 The entropy inequality

The results quoted in the previous section are not uniform with respect to ϵ as this parameter tends to zero. The core of the vanishing viscosity method is to recover some uniformity in such a way to be allowed to extract a subsequence, still denoted $(u^\epsilon)_{\epsilon\to 0+}$, converging boundedly almost-everywhere (in short, $b.a.e$). Then the dominated convergence theorem allows us to pass to the limit in integrals like

$$\int\!\!\int_{\mathbb{R}\times\mathbb{R}^+} g(u^\epsilon)\phi \, dxdt,$$

as soon as $g \in \mathcal{C}^0(I\!\!R^n)$ and $\phi \in L^\infty$ has a compact support.

Let us apply this principle to (1.12). We first integrates it against a non-negative test function $\phi \in \mathcal{D}(I\!\!R^2)$:

$$\int\int_{I\!\!R^2} \left(\eta(u^\epsilon)\phi_t + q(u^\epsilon)\phi_x - \epsilon(D^2\eta B(u^\epsilon)u^\epsilon_x|\phi_x)\right) dxdt + \int_{I\!\!R} \phi(x,0)\eta(u_0(x))dx$$
$$\geq \epsilon \int\int_{I\!\!R^2} \alpha(u^\epsilon)\|B(u^\epsilon)u^\epsilon_x\|^2\phi\, dxdt.$$

Thanks to the pointwise boundedness of u^ϵ, the integral of the two first terms remain bounded. Next the Cauchy-Schwarz inequality allows us to estimate the third term in terms of the right-hand side. Finally, we obtain an L^2-estimate of u^ϵ_x : for each bounded domain Ω in $I\!\!R \times I\!\!R^+$,

$$\epsilon \int\int_\Omega \|u^\epsilon_x\|^2 dxdt$$

remains bounded.

We now pass to the limit in the above inequality. For this, we first bound the right-hand side by below, by zero. Thanks to the convergence $b.a.e$, the integrals of the two first terms converge to the integrals of the limits. The third one is again bounded by

$$C\epsilon\|u^\epsilon_x\|_{L^2(\Omega)^\cdot},$$

which tends to zero as $\sqrt{\epsilon}$, thanks to the previous estimate. We thus obtain the *entropy inequality*

$$\int\int (\eta(u)\phi_t + q(u)\phi)\ dxdt + \int_{I\!\!R} \phi(x,0)\eta \circ a\, dx \geq 0. \tag{1.14}$$

for all non-negative test function ϕ.

We emphasize that this calculus may be carried out for the special case $\eta_\pm(u) := \pm u$, which yields to the **equality**

$$\int\int (u\phi_t + f(u)\phi_x)\ dxdt + \int_{I\!\!R} \phi(x,0)a\, dx = 0, \tag{1.15}$$

for all test function ϕ, since it may be written as the difference of non-negative test functions. The formula (1.15) means that u is a weak solution of the Cauchy problem (1.1,1.2). In particular it satasfies (1.1) in the distributional sense. Similarly, (1.14) implies that

$$\partial_t\eta(u) + \partial_x q(u) \leq 0, \tag{1.16}$$

in the distributional sense.

1.8 The entropy criterion

One says that a weak solution of (1.1) satisfies the *Lax' entropy criterion* if it satisfies the entropy inequality (1.14) for every entropy-flux pairs (η, q) with a convex η, or at least for one such pair having a physical meaning. Thus there is some room in this definition, depending on the size of the set of convex entropies under consideration. The previous calculus suggests that this set consists of all convex entropies for which $D^2\eta B$ induces a non-negative quadratic form, where B is the diffusion naturally given by the physics.

Let us emphasize that the entropy criterion with only one convex entropy may be two weak in realistic examples. That is, it leaves some non-uniqueness for the Cauchy problem. This happens even in the simplest case of a scalar conservation law, when the flux f is neither convex nor concave (see the next section). However, an elementary calculus using one convex entropy shows that the uniqueness holds provided there is a classical solution. Thus the non-uniqueness is due only to the appearance of discontinuities.

The entropy criterion does not impose any restriction to classical solutions, since these satisfy $\partial_t\eta(u)+\partial_x q(u)=0$. Similarly, it is not a restriction on domains where u is continuous and piecewise-Lipschitz. However, it induces irreversibility along discontinuities. With the same notation than in the Rankine-Hugoniot condition, it rewrites

$$[q(u)] \leq \frac{dX}{dt}[\eta(u)]. \tag{1.17}$$

Thus, discontinuous solutions are usually not reversible (that is $v(x,t):=u(-x,-t)$ is not admissible), except in the special case where the inequality (1.17) is an equality. This special case is unlikely because the non-linear algebraic system (with given a)

$$f(b)-f(a)=\sigma(b-a), \quad q(b)-q(a)=\sigma(b-a)$$

consists of $n+1$ equations with $n+1$ unknowns, thus solutions close to the trivial ones ($b=a$) coincide with them.

There is however an exception to this general remark, a frequent one because the systems given by continuum mechanics are far to be generic. This is encountered when a characteristic field (λ_p,r_p) is LD. Then, given an integral curve Γ of the eigenfield r_p, λ_p is constant along Γ and any two points (a,b) on Γ satisfy the Rankine-Hugoniot condition, plus the entropy equality, with $dX/dt=\lambda_p|_\Gamma$. Such reversible patterns are called *contact discontinuities*.

The opposite case of a GNL field (λ_p,r_p) gives rise to the following caracterisation of weak admissible discontinuities :

Theorem 1.8.1 *Let the p-characteristic field be GNL at b and let $r_p(b)$ be normalized. Then, on $H_p(b)\cap B(b;\epsilon)$, the states c located on the side $-r_p(b)$ are right states of a shock $(b,c;\sigma)$ satisfying the entropy inequality, whereas the states c located on the side $+r_p(b)$ are left states of a shock $(c,b;\sigma)$ which satisfies the entropy inequality. More precisely :*

$$q(c)-q(b)-\sigma(\eta(c)-\eta(b))=\frac{s^3}{12}D^2\eta_b(r_p,r_p)+\mathcal{O}(s^4),$$

where $c-b\sim sr_p(b)$, s being small.

Comparing this result with theorem 1.5.2, we conclude that the Lax's shock inequality and the entropy inequality select the same shocks, provided these are weak and correspond to GNL fields. However, this equivalence may fail for fields which are neither GNL nor LD, or for strong discontinuities.

1.9 The Oleinik/Liu criterion

For a scalar conservation law, every $\mathcal{C}^1$ function η is an entropy, whose flux q is given by $q'=f'\eta'$. Indeed, $\mathcal{C}^0$-entropies may be considered, thanks to the identity $(q-f'\eta)'=-f''\eta$.

Moreover, only one type of dissipation is available : B a positive function. Thus the entropy inequality must be written for all convex entropies, that is all convex functions. Now, the set of convex functions is the convex hull of affine functions and the so-called "Kruzkhov entropies" :

$$\eta_k(u):=|u-k|.$$

Thus a discontinuity is admissible if and only if it satisfies the Rankine-Hugoniot plus the entropy inequalities for all η_k. A straightforward analysis gives rise to the following equivalent property, called the **Oleinik criterion**.

Theorem 1.9.1 *Let (1.1) be a scalar equation and let $(a, b; \sigma)$ satisfy the Rankine-Hugoniot condition.*

If $a < b$, then $(a, b; \sigma)$ satisfies the entropy criterion if and only if the segment with ends $(a, f(a))$ and $(b, f(b))$ lies below the graph of f.

If $a > b$, then $(a, b; \sigma)$ satisfies the entropy criterion if and only if the segment with ends $(a, f(a))$ and $(b, f(b))$ lies above the graph of f.

Let us point that the slope of this segment is σ. One easily see that the Oleinik criterion implies the Lax shock inequality. But the converse is not true in general, except (as mentioned above) if f is either convex or concave. Its powerness may be seen in the existence and uniqueness theorem of Kruzhkov, which states that the Cauchy problem for a scalar equation is well-posed in the class of bounded measurable functions, when considering the entropy criterion.

There is no counterpart of this theory for systems (that is when $n \geq 2$). The Lax shock inequality still makes sense, as well as the entropy criterion. However, none of them implies (apparently) the well-posedness of the Cauchy problem. In particular, since there are much fewer entropies for systems than scalar equations, the inequalities (1.17) do not restrict enough the the set of admissible discontinuities. There have been attempts to extend the Oleinik criterion, involving only algebraic quantities as the shock velocity $\sigma(a, b)$ and the eigenvalues λ_p. Here is the so-called *E-criterion*, due to Hsiao Ling and Tong Zhang [4] and Tai-Ping Liu [6, 7].

Let us first fix some notations. One denotes by $\sigma(a, b)$ the velocity of the discontinuity $a \mapsto b$ whenever $b \in H(a)$ (or equivalently $a \in H(b)$). Thus we have

$$f(b) - f(a) = \sigma(a, b)(b - a).$$

Next, we assume that $H(a) \setminus \{a, b\}$ has a connected component $S(a)$, which displays the properties :

- $S(a)$ is an arc of a Jordan $\mathcal{C}^1$-curve,
- $S(a)$ ends at a and b.

Thanks to the theorem 1.5.1, $S(a)$ is tangent at a to some eigenvector $r_p(a)$. We thus let write $S_p(a)$ instead of $S(a)$. We similarly assume that $H(b) \setminus \{a, b\}$ has a component $S'(b)$, which is a smooth arc from a to b. As before, $S(b)$ must be a $S_q(b)$ for some index q.

Definition 1.9.1 *The discontinuity is said to satisfy the* (E) criterion *if*

1. $p = q$,
2. $\sigma(a, b) \leq \sigma(a, u)$, *for all u in $S_p(a)$,*
3. $\sigma(u, b) \leq \sigma(a, b)$, *for all u in $S_q(b)$.*

Such a discontinuity is called an admissible p-shock wave, with respect to the (E) criterion. Letting u tend to a or b, we find that such p-shocks are also p-shocks in the sense of Lax : $\lambda_p(b) \leq \sigma(a, b) \leq \lambda_p(a)$. It is not difficult to check that, in the scalar case, each inequality in the definition is equivalent to the Oleinik condition. In this case, the "curves" $S_p(a)$ and $S_q(b)$ coincide with the segment (a, b) or (b, a). More generally, for weak shocks (that is $\|b-a\| << 1$), the two conditions 2 and 3 are equivalent to each other (actually, the condition $p=q$ becomes trivial, because $\sigma(a,b)$ must be close to the spectra of $df(a)$ and of $df(b)$) : for if c is a point of $S_p(a)$ where $\sigma(a,c)=\sigma(a,b)$, then $f(b)-f(a)=\sigma(b-a)$ and $f(c)-f(a)=\sigma(c-a)$ imply $f(b)-f(c)=\sigma(b-c)$, that is $c\in H_q(b)$ and $\sigma(b,c)=\sigma(a,b)$. Since c,b,a are close to each other, a bifurcation analysis (see section 2.1) show that actually $c\in S_q(b)$. Then oscillations of $u\mapsto\sigma(a,u)$ along $S_p(a)$ correspond to oscillations of $u\mapsto\sigma(u,b)$ along $S_q(b)$.

On the contrary, we shall see in the next chapter that some strong shocks must be admitted although they do not satisfy the (E) criterion. Thus the (E) criterion makes sense only for weak shocks.

Chapter 2

Viscous shock profiles

In all this chapter, we make use of the viscous approximation (0.2) in order to select admissible shocks. The main idea is that a shock $(u_l, u_r; s)$, which is a rough discontinuity at the macroscopic level (that is when $\epsilon = 0$), has actually an internal structure at the scale ϵ. This means that the left state u_l and the right state u_r (at a given discontinuity point (x_0, t_0)) are connected by a travelling wave

$$U\left(\frac{x - x_0 - s(t - t_0)}{\epsilon}\right),$$

up to a small and smooth correction. Thus U will solve (0.2) up to a bounded correction. Expanding in powers of ϵ, we immediately find a differential equation

$$(B(U)U')' = (f(U) - sU)'.$$

This may be integrated once, to give $B(U)U' = f(U) - sU - r$, where r is a constant vector. The matching $U(-\infty) = u_l$, $U(+\infty) = u_r$ suggests the choice $r = f(u_r) - su_r = f(u_l) - su_l$, which makes sense, because of the Rankine-Hugoniot condition. The function $\xi \mapsto U(\xi)$ is called the *profile* of the shock $(u_l, u_r; s)$. We thus obtain the *profile equation*

$$B(U)U' = f(U) - f(u_l) - s(U - u_l). \tag{2.1}$$

Definition 2.0.2 *The discontinuity $(u_l, u_r; s)$ is said to satisfy the* viscosity criterion *if there exists a solution U of (2.1) with the properties $U(-\infty) = u_l$, $U(+\infty) = u_r$.*

In the geometric language, such a U is a heteroclinic connection from u_l to u_r, which both are stationary points of (2.1). One easily checks that the viscosity criterion implies the entropy criterion, since the profile is an exact solution of (0.2), which converges *b.a.e* towards the discontinuity between u_l and u_r. Actually, a direct calculation using the profile equation gives the same result.

A complete theory of viscous profiles and admissible shocks is beyond any attempt, because of the strong non-linearity of (2.1). In realistic applications, B is even not invertible, so that (2.1) is not an ODE but rather an algebraic-differential equation. Since the rank of B may depend on the examples under consideration, and also on the approximations made in the modelling, too many cases occur. Much more, the study of (2.1) for magnehydrodynamics shows that the patterns do not only depend on the rank of B, but also on the ratio of various diffusion coefficients. For all these reasons, we shall present here a simplified theory, where B is assumed to be invertible (!). Then (2.1) reduces to a genuine ODE

$$U' = G(U; s, u_l), \quad G(u; s, a) := B(u)^{-1}(f(u) - f(a) - s(u - a)). \tag{2.2}$$

2.1 Viscous profiles for weak shocks

Weak shocks are those for which u_r is closed to u_l. We look for small orbits between both states, although large ones might exist, because large ones are irrelevant in the modelling. Then the existence of an orbit for (2.2) may be studied by a bifurcation analysis. We first extend this system as an ODE in a neighbourhood of $P = (u_l, \lambda_p(u_l))$ in $I\!R^{n+1}$:

$$\begin{pmatrix} U \\ s \end{pmatrix}' = g\begin{pmatrix} U \\ s \end{pmatrix}; u_l\Bigg) = \begin{pmatrix} G(U; s, u_l) \\ 0 \end{pmatrix}. \tag{2.3}$$

We first remark that $\mu = 0$ is a double eigenvalue of the Jacobian matrix dg at P. We next claim that $dg(P)$ does not have any other purely imaginary eigenvalue (one uses the inequality (1.11)). Thus the local theory of dynamical systems (see [11]) ensures the existence of a *center manifold* $\mathcal{M}$ in the vicinity of P, of dimension two (the multiplicity of μ). This manifold, which is as smooth as g, is invariant by the flow of (2.2) and contains all those orbits which remain forever in a vicinity of P. In particular, it contains all the rest points and all the heteroclinic orbits which are close to P.

Thanks to the theorem 1.5.1, the rest points belong to two curves which intersect transversally at P : the curve $L_l := \{u_l\} \times I\!R$ and the curve $\mathcal{H}_p(u_l) := \{(b, \sigma(u_l, b)); b \in H_p(u_l)\}$.

Last, the orbits on $\mathcal{M}$ are weared by the straight lines s =constant. These lines meet L_l transversally and the direction of the flow changes as we cross L_l. It also changes as we pass from one side of P to the other one. In other words, (u_l, s) is attractive for $s > \lambda_p(u_l)$, repulsive for $s < \lambda_p(u_l)$. Thus admissible shock waves are those for which

- $\sigma(u_l, u_r) \leq \lambda_p(u_l)$,
- $(u_r, \sigma(u_l, u_r))$ is the first rest point (which means that $u_r \in H_p(u_l)$), beyond u_l on the line $s = \sigma(u_l, u_r)$.

This is nothing but the first inequality of the (E) criterion, which explains why this condition was introduced. We thus have :

Theorem 2.1.1 *A weak shock wave admits a viscous profile if and only if it satisfies the (E) criterion.*

The reason why this theorem cannot be extended to strong shocks is that the center manifold cannot be defined globally[1]. Let us now describe the most common situations.

In the case of a GNL field (λ_p, r_p), one checks that $\mathcal{H}_p(u_l)$ intersects transversally the line $s = \lambda_p(u_l)$ at P, so that $(u_l, u_r; \sigma(u_l, u_r))$ is admissible if and only if $\lambda_p(u_l) > \sigma(u_l, u_r)$, that is the shock is a Lax one. Moreover, given a number $s \sim \lambda_p(u_l) - 0$, there is one and only one admissible shock of the form $(u_l, u_r; s)$. Of course, this characterization holds only for weak shocks.

In the marginal case where $d\lambda_p \cdot r_p$ vanishes at u_l, but $d(d\lambda_p \cdot r_p) \cdot r_p$ does not, again the admissible shocks are those for which $\lambda_p(u_l) > \sigma(u_l, u_r)$. But now, given $s \sim \lambda_p(u_l) - 0$, there are two admissible shocks $(u_l, u_r; s)$ or none, depending on the sign of $d(d\lambda_p \cdot r_p) \cdot r_p$ at u_l.

[1]It would require that the function g be globally lipschitz, with a small enough lipschitz constant. This is unlikely.

2.2 Examples in gas dynamics

A realistic application, as gas dynamics, requires to deal with a singular diffusion matrix, so that the analysis of the section 2.1 does not apply.

For the sake of simplicity, we consider the plane waves of a compressible fluid in Lagrangian coordinates : x has the dimension of a mass, and each particle moves along a line $x =$ constant. In the absence of viscosity and heat conduction, the system writes

$$\left\{\begin{array}{rcl} v_t & = & z_x, \\ z_t + p_x & = & 0, \\ (e + \frac{1}{2}z^2)_t + (pz)_x & = & 0. \end{array}\right. \tag{2.4}$$

Hereabove, v is the specific volume, that is $1/v$ is the mass density, z is the velocity, e is the specific internal energy and p the pressure. They are related by the state law $(v,e) \mapsto p = p(v,e)$. We shall assume the perfect gas law : $pv = (\gamma - 1)e$, where $\gamma \in]1, 5/3]$ is a constant.

The study of the Rankine-Hugoniot condition gives rise to two types of discontinuities :

- Contact discontinuities, for which $\sigma = 0$, $[z] = 0$ and $[p] = 0$,
- Propagating discontinuities, for which $\sigma \neq 0$ and

$$e_r - e_l + \frac{p_r + p_l}{2}(v_r - v_l) = 0.$$

There are several different cases in the analysis of profiles, depending on which diffusion processes we take in account : heat conduction, viscosity or both.

2.2.1 Profiles with heat conduction

The heat conduction (Fourier law) changes the system to

$$\left\{\begin{array}{rcl} v_t & = & z_x, \\ z_t + p_x & = & 0, \\ (e + \frac{1}{2}z^2)_t + (pz)_x & = & \epsilon(k(v,e)\theta_x)_x, \end{array}\right. \tag{2.5}$$

where $k > 0$ is a given function. For a perfect gas, the temperature θ is a constant multiple of e and we may fix $\theta = e$. Then profiles are such that $z + \sigma v \equiv z_l + \sigma v_l$, $p - \sigma z \equiv p_l - \sigma z_l$ and

$$ke' = pz - p_l z_l - \sigma\left(e + \frac{1}{2}z^2 - e_l - \frac{1}{2}z_l^2\right). \tag{2.6}$$

Eliminating p and z, we see that the right hand side of (2.6) is a quadratic polynomial of v, which must vanishes at v_l and v_r. Thus it is $\alpha(v - v_l)(v - v_r)$ for some α. On the other hand,

$$(\gamma - 1)e' = p'v + pv' = (p - \sigma^2 v)v'.$$

Finally, the profile equation reduces to an ODE with respect to v :

$$k(p_l + \sigma^2 v_l - 2\sigma^2 v)v' = \sigma^3 \frac{\gamma + 1}{2}(v - v_l)(v - v_r). \tag{2.7}$$

This equation degenerates at the point v^* defined by

$$v^* := \frac{1}{2\sigma^2}(p_l + \sigma^2 v_l).$$

This singularity prevents a profile to cross v^*. Thus the existence of the profile is subjected to the necessary and sufficient condition $(v^* - v_l)(v^* - v_r) > 0$, that is

$$(p_l - \sigma^2 v_l)(p_r - \sigma^2 v_r) > 0,$$

which amounts to

$$\frac{\gamma+1}{3\gamma-1} < \frac{v_l}{v_r} < \frac{3\gamma-1}{\gamma+1}.$$

This restriction is a severe one since one usually admits shock waves[2] such that

$$\frac{\gamma-1}{\gamma+1} < \frac{v_l}{v_r} < \frac{\gamma+1}{\gamma-1},$$

and

$$\left(\frac{\gamma+1}{3\gamma-1}, \frac{3\gamma-1}{\gamma+1}\right) \subset \left(\frac{\gamma-1}{\gamma+1}, \frac{\gamma+1}{\gamma-1}\right).$$

Thus the lack of newtonian viscosity has a bad effect on the admissibility criterion.

2.2.2 Profiles with newtonian viscosity

With newtonian viscosity, the system takes the form

$$\left\{\begin{array}{rcl} v_t & = & z_x, \\ z_t + p_x & = & \epsilon(b(v,e)z_x)_x, \\ (e + \frac{1}{2}z^2)_t + (pz)_x & = & \epsilon(bzz_x)_x, \end{array}\right. \tag{2.8}$$

where $b > 0$ is a given function.

A strange feature of this approximation is that it is compatible with contact discontinuities : these admit discontinuous profiles.

Regarding propagating shocks, the profiles obey to $z + \sigma v = z_l + \sigma v_l$, plus two differential equations

$$\begin{array}{rcl} bz' & = & p - \sigma z - p_l + \sigma z_l, \\ bzz' & = & pz - p_l z_l - \sigma\left(e + \frac{1}{2}z^2 - e_l - \frac{1}{2}z_l^2\right). \end{array}$$

Elimination gives rise to only one ODE in the variable z :

$$vbz' = \frac{\gamma-1}{2}(z - z_l)(z - z_r).$$

Since v is an affine function of z, it varies between v_l and v_r, thus remains positive. We deduce that this equation has always a heteroclinic orbit between z_l and z_r. Finally, the Lax shock inequality is equivalent to the viscosity criterion. It is actually equivalent to the (E) condition too.

As a conclusion, the best approximation (without profiles for contact discontinuities, with profiles for all physical shocks) requires both heat conduction and newtonian viscosity. But it gives rise to an ODE in the plane of coordinates (e, z). The existence of connecting orbits is now a more difficult problem.

[2] and we need them in order to solve the Riemann problem.

2.3 Structural stability of heteroclinic orbits

The existence of a viscous profile might be considered as unsufficient in applications, if the profile does not persist under small disturbances of either of the states u_l, u_r or the flux f, just because we never know these data with perfect accuracy. For this reason, we must add to our criterion a stability assumption. We ask the admissible shocks to admit a *structurally stable* profiles. This notion has been extensively studied in dynamical systems theory. Here is a caracterization :

Proposition 2.3.1 *A heteroclinic orbit of a dynamical system $u' = g(u)$, from a to b, is structurally stable if and only if*

- *the points a and b are hyperbolic rest points of g, that is $dg(a)$ and $dg(b)$ do not admit pure imaginary eigenvalues,*
- *the stable manifold $W^s(b)$ at b and the unstable manifold $W^u(a)$ at a intersect transversely along the orbit. In other words, their tangent subspaces satisfy*

$$T_c W^s(b) \oplus T_c W^u(a) = \mathbb{R}^n$$

(this property does not depend on the point c we choose on the orbit).

Since these tangent spaces both contain the direction $g(c)$, the dimension of their sum is strictly less than the sum of their dimensions. Structural stability thus need the sum of dimensions to be at least $n+1$. This means that the number of eigenvalues of $dg(a)$ with positive real part, plus the number of eigenvalues of $dg(b)$ with negative real part, is at least $n+1$. Applying these principles to viscous profiles, one obtains the following necessary conditions for structural stability :

- $\sigma(a,b)$ is not equal to one of the eigenvalues of $dg(a)$ and $dg(b)$,
- there is an index p such that $\lambda_p(b) < \sigma(a,b) < \lambda_p(a)$.

Let us observe that weak shock waves satisfying the (E) condition always admit a structurally stable profile, except in marginal cases where $\sigma(a,b)$ equals either $\lambda_p(a)$ or $\lambda_p(b)$. In particular, Lax's shocks satisfy the dimensional necessary condition, the weak Lax's shocks being structurally stable.

2.4 Under-compressive shock waves

Under-compressive shocks are those for which

$$\lambda_p(a), \lambda_p(b) < \sigma(a,b) < \lambda_{p+1}(a), \lambda_{p+1}(b) \tag{2.9}$$

holds. It does not make sense to call them p-shocks rather than $(p+1)$-shocks. One would merely use the odd terminology $(p+\frac{1}{2})$-shock. The existence of a heteroclinic orbit implies that $W^s(b)$ and $W^u(a)$ be non-trivial, that is $dg(a)$ and $dg(b)$ have at least one eigenvalue with negative (resp. positive) real part. One thus obtain the restriction $1 \leq p \leq n$.

For instance, only one kind of such transitional waves have to be considered when $n = 2$, namely $p = 1$. In such a case, both a and b are saddle points of the vector field g.

Under-compressive shock waves usually do not admit a viscous profile, since such profiles cannot be structurally stable, according to proposition 2.3.1. Such orbits do not persist under generic perturbations depending on one parameter. However, they may persit under perturbations which run over a codimension-one set. Roughly speaking, let (a_0, b_0) and B_0 be such that a profile exists from a_0 to b_0. Then the set of triples $(a, b; B)$ such that a profile exists from a to b is of codimension n (the Rankine-Hugoniot condition is responsible for a codimension $n-1$) in the neighbourhood of $(a_0, b_0; B_0)$, under a generic condition. This condition is the non-vanishing of a so-called "Melnikov integral" along the profile. In most cases, and for a given viscous tensor B, we may parametrize the set of admissible undercompressive shocks (a, b) by the left state a only, or either by the right state b. For instance, the left state a determines both $b = \beta(a)$ and the shock velocity $s(a) := \sigma(a, \beta(a))$. Let us remind that in the case of Lax shocks, a and $\sigma(a, b)$ are independant of each other and determine b through the Rankine-Hugoniot condition.

Let us show how under-compressive shocks may be used in the Riemann problem. We consider the 2×2 case, and choose an admissible under-compressive shock (a, b). Now let (u^l, u^r) be the Riemann data, chosen close to (a, b). Because of strict hyperbolicity, there are wave curves originated at u^l or passing through u^r. The Riemann problem will be solved using a $1 - wave$ from u^l to an intermediate state u_1, still close to a. One has a parametrization $u_1 = \phi_1(u^l; s_1)$, where $s_1 \sim \lambda_1(u^l)$. Next one jumps from u_1 to a point u_2 by an admissible under-compressive shock : $u_2 = \beta(u_1)$. Last, one needs a 2-wave from u_2 to u^r, which writes $u^r = \phi_2(u_2; s_2)$, with $s_2 \sim \lambda_2(u^r)$. The solution is found by solving the non-linear system

$$\phi_2(\beta(\phi_1(u^l; s_1)); s_2) = u^r. \tag{2.10}$$

This system consists of two scalar equations (because $n = 2$) and two scalar unknowns s_1, s_2. When $(u^l, u^r) = (a, b)$, it admits the trivial solution $s = (\lambda_1(a), \lambda_2(b))$. Thus the Riemann problem admits a unique solution for (u^r, u^l) close to (a, b), provided a generic condition holds (one applies the implicit function theorem). Let us point out that since $\lambda_1(a) < \sigma(a, b) < \lambda_2(b)$, the waves involved by this solution may be glued together. Since also $\lambda_1(b) < \sigma(a, b) < \lambda_2(a)$, it is clear that we might not use any other kind of wave.

The abovementioned sufficient condition is that the differential of the left hand side of (2.10) with respect to s is invertible when $(u^l, u^r, s) = (a, b, \lambda_1(a), \lambda_2(b))$. It thus writes

$$\det(d\beta(a) r_1(a), r_2(b)) \neq 0,$$

or equivalently

$$l_2(b) \cdot d\beta(a) r_1(a) \neq 0.$$

We know turn to the local solvability of the Cauchy problem in the spirit of the section 1.4. With the same notations, we now assume that $(a^-(0), a^+(0); \sigma_0)$ is an admissible under-compressive shock. In step 2, we now have $q + r = n$, instead of $n - 1$. The counterpart is that the Rankine-Hugoniot condition gives $n-1$ boundary condition, while the law $a \mapsto \sigma(a, b)$ gives an extra one. Finally, the Cauchy problem is well-posed in spaces of piecewise-smooth function, as proved by Freistühler [2] :

Theorem 2.4.1 *Let $a^\pm : I\!R^\pm \to \mathcal{U}$ be $\mathcal{C}^1$ functions and let $a \in L^\infty(I\!R)$ be the function whose restrictions to $I\!R^\pm$ are $a^\pm$. We denote by $u^\pm$ the states $a^\pm(0)$ and we suppose that $(u^-, u^+; \sigma)$ is an admissible under-compressive shock :*

$$\lambda_p(u^\pm) < \sigma < \lambda_{p+1}(u^\pm), \quad u^+ = \beta(u^-).$$

We denote the shock velocity $\sigma(u, \beta(u))$ by $S(u)$.

We now assume the compatibility conditions :

$\mathcal{C}^0$ ***compatibility***

$$\exists \sigma \in I\!R, \quad [f \circ a] = \sigma [a].$$

$\mathcal{C}^1$ ***compatibility***

$$[(df \circ a - \sigma)^2 a'] = \left(dS(u^-)(df(u^-) - \sigma) a'(0-) \right) (u^+ - u^-).$$

Finally, let us assume the (generic) condition :

$$\det(R_1(u^-), \cdots, R_p(u^-), r_p(u^+), \cdots, r_n(u^+)) \neq 0,$$

where $R_k(u^-) := (\lambda_k(u^-) - \sigma) r_k(u^-) + (dS(u^-) \cdot r_k(u^-))(u^+ - u^-)$.

Then the Cauchy problem admits one and only one piecewise $\mathcal{C}^1$ *solution, with a discontinuity located on a* $\mathcal{C}^2$ *curve originated at* $(x,t) = (0,0)$. *locally in time.*

Chapter 3

Discrete shock profiles

We consider in this chapter approximate solutions given by numerical analysis. We a priori choose a uniform grid in $I\!R \times I\!R^+$. The mesh points are $(j\Delta x, m\Delta t)$, where (j,m) runs over $Z\!\!Z \times I\!N$. The mesh sizes $\Delta x, \Delta t$ are small positive constants. The solution $u(x,t)$ of (1.1) is approached by discrete values u_j^m. We consider here explicit difference schemes, in the general form

$$u_j^{m+1} = G(u_{j-p}^m, \cdots, u_{j+q}^m).$$

Such schemes are called $(p+q+1)$-points schemes. We then reconstruct an approximate solution from these values by any stable extrapolation method. We expect that convenient hypotheses will imply the convergence of $U^{\Delta x, \Delta t}$ as $\Delta x, \Delta t$ go to zero.

Let us first observe that if the initial data assumes a constant value u^l for $x < 0$, then the exact solution will be equal to u^l for $x < \Lambda_1 t$, where Λ_1 is the lower bound of $\lambda_1(u(x,t))$ for all (x,t). Besides, the approximate solution equals u^l for $j < -qm$, that is when $x < -qt\sigma$, where σ denotes the ratio $\Delta x/\Delta t$; having the dimension of a velocity. Similarly, an initial data constant for $x > 0$ yields an approximate solution being constant for $x > pt\sigma$. Thus the signals propagate at velocities between $-q\sigma$ and $p\sigma$, for the approximate solution, but between $\lambda_1(u)$ and $\lambda_n(u)$ for the exact solution. The consistency of the scheme needs that the approximate signals propagate at least as fast as the exact ones, which means that in the limit, when $(\Delta t, \Delta x)$ tends to zero :

$$-q \lim \sigma \leq \inf_u \lambda_1(u), \quad \sup_u \lambda_n(u) \leq p \lim \sigma, \tag{3.1}$$

where the range of u contains the values achieved in the approximation and/or by the exact solution.

In practice, one keeps fixed the ratio σ or one ajust it at each time step by considering the values u_j^m taken at this time. The inequalities (3.1) are refered to the "Courant-Friedichs-Levy" condition. It often appears naturally in the stability analysis of the scheme, when one linearizes around a constant state.

The simplest schemes are those for which $0 \leq p, q \leq 1$. The case $q = 0$ is called upwind scheme and requires that $\lambda_1 \geq 0$. Since most systems of conservation laws have eigenspeeds of both signs, one frequently encounters the case $p = q = 1$, with the CFL condition $\sup_u \rho(df(u)) \leq \sigma$.

3.1 Conservative difference schemes

One usuallys ask a scheme to reproduce faithfully the conservation property of the system. This requires the function G to be of the form

$$G(u_{-p},\dots,u_q) = u_0 + \frac{1}{\sigma}(F(u_{-p},\dots,u_{q-1}) - F(u_{p+1},\dots,u_q)).$$

The scheme is consistent provided the *numerical flux* F satisfies the property

$$F(a,\cdots,a) = f(a) \quad \forall a \in \mathcal{U}.$$

Let u be a $\mathcal{C}^1$-solution of (1.1) in some open set. Then

$$\begin{aligned}(3.2)\qquad & (u(x,t+\Delta t) - u(x,t))/\Delta t + \\ & \tfrac{1}{\Delta x}\Big(F(u(x-(p-1)\Delta x,t),\dots,u(x+q\Delta x,t)) - F(u(x-p\Delta x,t),\dots,u(x+(q-1)\Delta x,t))) \\ & = \mathcal{O}(\Delta x),\end{aligned}$$

that is the scheme is of order one, at least.

Let us point out that most of numerical fluxes also depend ond the ratio σ.

Here are examples of three-points schemes ($p = q = 1$) :

Lax-Friedrichs scheme : the flux is

$$F_{LF}(a,b) = \frac{1}{2}(f(a)+f(b)) + \frac{\sigma}{2}(a-b)$$

and the scheme writes

$$u_j^{m+1} = \frac{1}{2}(u_{j-1}^m + u_{j+1}^m) + \frac{1}{2\sigma}(f(u_{j-1}^m) - f(u_{j+1}^m)).$$

One may quote that the subgrids defined by $j+m$ odd or $j+m$ even do not interact to each other.

Centered finite differences : a naive scheme is

$$u_j^{m+1} = u_j^m + \frac{1}{2\sigma}(f(u_{j-1}^m) - f(u_{j+1}^m)),$$

which is simpler than the Lax-Friedrichs scheme and makes the odd and even sub-grids interact. However, both experiments and mathematical analysis show that this scheme is violently unstable. It should not be used in any computation.

Lax-Wendroff scheme : this is a second-order scheme, in the sense that $\mathcal{O}(\Delta x)$ must be replaced by $\mathcal{O}(\Delta x^2)$ in (3.2). The flux is

$$F_{LW} = \frac{1}{2}(f(a)+f(b)) + \frac{1}{2\sigma}df\left(\frac{a+b}{2}\right)(f(a)-f(b)).$$

Upwind scheme : here $q = 0$ and we assume that $\lambda_1 \geq 0$. The scheme writes

$$u_j^{m+1} = u_j^m + \frac{1}{\sigma}(f(u_{j-1}^m) - f(u_j^m)),$$

thus the flux is

$$F_U(a,b) = f(a).$$

Godunov scheme : apparently, the flux F_G is quite simple, since $F_G(a,b) = f(R(a,b))$ is a value of the flux of (1.1). However, the sampling function R is involved : $R(a,b)$ is the value, along the vertical axis $x=0$, of the self-similar solution $w_{a,b}$ of the Riemann problem with left state equal to a and right state equal to b. This sampling may be ambiguous if this solution is discontinuous accross $x=0$, but the value of F_G is not, because $f \circ w_{a,b}$ is continuous accross this axis, thanks to the Rankine-Hugoniot condition. The flux F_G is generally not a $\mathcal{C}^1$ function, when such discontinuities occur, that is when the sign of an eigenspeed may change.

The Godunov scheme may be viewed as the generalization of the upwind scheme for systems having eigenspeeds of both signs.

3.2 Linear stability of constants

The stability analysis around constant states leads to the following requirement, called Von Neumann's criterion

$$\rho(\mathcal{G}(a;\xi)) \leq 1, \quad \forall a \in \mathcal{U}, \forall \xi \in I\!R, \tag{3.3}$$

where

$$\mathcal{G}(a;\xi) := \sum_{-p}^{q} e^{ik\xi} \frac{\partial G}{\partial z_k}(a,\dots,a).$$

Let us remark that, because of conservativity, $\mu = 1$ always belongs to the spectrum of the matrix $\mathcal{G}(a;0) = I_n$. A straightforward calculus yields a second-order expansion of each eigenvalues of $\mathcal{G}(a;\xi)$, for small values of ξ :

$$\rho_p(a;\xi) = 1 - i\mu_p(a)\xi + \left(\frac{1}{2}\mu_p(a) + l_p(a) \left(\sum_{p}^{q-1} k \frac{\partial F}{\partial z_k} \right) r_p(a) \right) \xi^2 + \cdots,$$

where $\mu_p := \lambda_p/\sigma$. Since μ_p is real, the modulus of $\rho_p(a;\xi)$ is $1 + \mathcal{O}(\xi^2)$ and the condition (3.3) implies

$$\mu_p^2 + \mu_p + \frac{2}{\sigma} l_p \left(\sum_{p-1}^{q} k \frac{\partial F}{\partial z_k} \right) r_p \leq 0, \tag{3.4}$$

at each point of the domain $\mathcal{U}$. Hereabove the derivatives of F are computed at the points $(a,\dots,a)$.

When applied to various schemes, this necessary condition for stability reads as the CFL condition $\rho(df(a)) \leq \sigma$ in case of Lax-Friedrichs or Godunov schemes, whereas it is trivial for the Lax-Wendroff scheme. However, condition (3.3) yields to the CFL condition for the Lax-Wendroff scheme ; this does not come from (3.4) because $|\rho_p| = 1 + \mathcal{O}(\xi^4)$. The fact that the coefficient of ξ^2 vanishes means that the Lax-Wendroff scheme is second-order accurate.

3.3 Profiles

Next to constant states, shock waves are of exceptional interest. We wish that a difference scheme give a good approximation of such patterns. Mimicking the viscous approximation, we expect that, given a shock wave $(u^+, u^-; s)$, there is an exact solution of the difference scheme under consideration, in the form of a function of the travelling variable $x - st = j\Delta x - ms\Delta t =$

$(j-ms/\sigma)\Delta x$, with a convenient scaling. Clearly, Δx plays the rôle of the small scale, as ϵ did in the viscous profiles. Thus, u_j^m is expressed by means of a discrete profile v :

$$u_j^m = v(j-m\eta), \quad \eta := s/\sigma. \tag{3.5}$$

Pluging the formula (3.5) into the difference scheme gives the following functional equation :

$$v(y-\eta) = v(y) + \frac{1}{\sigma}(F(v(y-p),\dots,v(y+q-1)) - F(v(y-p+1),\dots,v(y+q))), \tag{3.6}$$

for all y of the form $j-\eta m$, $(j,m)\in \mathbb{Z}\times\mathbb{N}$.

An important feature is that the solvability of (3.6), with data $v(\pm\infty)=u^\pm$, depends on arithmetical properties of the dimensionless ratio η. It reduces to a finitely dimensional dynamical system if η is rational. Conversely, an irrational η gives rise to small divisors questions, so that the existence theory relies upon diophantine assumptions (see a preprint of Tai-Ping Liu & Shih-Hsien Yu, 1997). At first glance, one may observe that the set of values $j-m\eta$ is discrete if $\eta\in\mathbb{Q}$ (it is $\frac{1}{l}\mathbb{Z}$, where l is the denominator of η), whereas it is dense in $\mathbb{R}$ in the opposite case.

3.4 Existence when η is rational

Let $\eta = h/l$ be a rational number. Then, rescaling by a factor l, one writes (3.6) as

$$\begin{aligned} v(k-h) &= v(k) + \frac{1}{\sigma}(F(v(k-pl),\dots,v(k+(q-1)l)) \\ &\quad -F(v(k-(p-1)l),\dots,v(k+ql))), \end{aligned}$$

which may be viewed as a discrete dynamical system whenever F is invertible with respect to its first argument and w.r.to its last one (let us note that $|h|<\max(p,q)$ because of the CFL condition if this one is strict).

Inverting F, one rewrites (3.7) as a dynamical system with $N=(p+q+1)l-1$ steps,

$$v(k+ql) = \mathcal{F}(v(k-pl),\dots,v(k+ql-1)),$$

that is a dynamical system in $\mathbb{R}^{nN}$. The existence of heteroclinic orbits (defining discrete shock profiles) has been considered by Majda and Ralston [9], for small amplitude shock waves. The proof uses a bifurcation analysis and a center manifold argument, as in a viscous case. It is however more technical, since the ambiant space has dimension nN and the center manifold has dimension n instead of 2. Furthermore, one can make use of a discrete translational invariance rather than a continuous one. For all these reasons, the existence of profiles has been proved so far with the following restriction :

- The ratio $\|u^+-u^-\|/l$ is small (thus we cannot expect to attack the irrational case by a density argument),
- The scheme is "non-resonnant", that is $\rho(\mathcal{G}(a;\xi))<1$ whenever $\xi\notin 2\pi\mathbb{Z}$. This excludes a priori the Lax-Friedrichs scheme, but this technical difficulty may be overcome as follows : the double iteration map is still a three-points scheme (because of the uncoupling of odd and even grids), which is non-resonant is the CFL inequality is strict.

A simplified proof of Majda-Ralston theorem may be found in the Habilitation thesis of S. Benzoni (see http://www.umpa.ens-lyon/˜benzoni/hdr.ps).

3.5 The irrational case

Herebelow is a general result concerning profiles. We first define a vector-valued function $h \mapsto V(h)$ by

$$V(h) := \sum_{j \in \mathbb{Z}} (v(j+h) - v(j)).$$

This definition is meaningfull in most cases, because of the convergence of v towards its limits $u^{\pm}$. For instance, in the most common case where v converges exponentially fast to $u^{\pm}$, the sum converges. This is still true if v has an algebraic decay at infinity ($v(y) = u^{\pm} + a_{\pm}/y + b_{\pm}/y^2 + \ldots$), since then $v(j+h) - v(j)$ will be an $\mathcal{O}(j^{-2})$.

Proposition 3.5.1 *Let h belong to the subgroup $\mathbb{Z} + \eta\mathbb{Z}$. Then $V(h) = h(u^+ - u^-)$.*

Proof

The formula is clear if $h \in \mathbb{Z}$, because of reordering (here one does not use the fact that v is a discrete profile). Actually, one even have $V(h+y) = V(y) + h(u^+ - u^-)$ when $h \in \mathbb{Z}$.

Thus it remains the case of $h = m\eta$, $m \in \mathbb{Z}$. But a use of (3.6) with $y = z + j + \eta$ gives

$$V(z+\eta) - V(z) = \frac{1}{\sigma} \sum_{j \in \mathbb{Z}} (\mathcal{F}(j+1) - \mathcal{F}(j)),$$

where $\mathcal{F}$ has limits $f(u^{\pm})$ at $\pm\infty$. Therefore,

$$V(z+\eta) - V(z) = \frac{1}{\sigma}(f(u^+) - f(u^-)) = \eta(u^+ - u^-),$$

thanks to the Rankine-Hugoniot condition. An induction step then gives the conclusion. ■

Clearly, the proposition 3.5.1 does not depend on the rationality of η. But in the irrational case, the subgroup $\mathbb{Z} + \eta\mathbb{Z}$ is dense in $\mathbb{R}$. Thus the existence of sufficiently smooth profile would imply the continuity of V and therefore $V(h) \equiv h(u^+ - u^-)$ for every real number h. Since irrational numbers are dense within the real numbers, we may expect that this equality extends to rational η, provided the profile exists for all η (constrained by the CFL condition) and depends continuously on η with values in some suitable functional space.

It has been shown however (see [10]) that this equality does not hold for the Godunov scheme as $\eta = 0$ (steady shocks), for the full gas dynamics. Since this scheme and this system do not display any pathology, we conjecture that the equality $V(h) \equiv h(u^+ - u^-)$ fails to hold as η is rational in most cases. Therefore, something is wrong in the previous analysis and we may conclude that

- Either the profile does not exist for certain irrational η,
- Or it decays too slowly or it is not smooth enough as $y \to \pm\infty$,
- Or it does not depend in a smooth enough way on η.

Let however mention the following result, which leaves some hope for the scalar case, the p-system, the isentropic gas dynamics or, last, the full gas dynamics with $\gamma = 3$:

Proposition 3.5.2 *Let the flux f have $n-1$ linear components. Let also consider the Lax-Friedrichs scheme. Let finally assume that $\eta = 0$. Then all the discrete profiles satisfy the identity $V(h) \equiv h(u^+ - u^-)$.*

Proof
The profile equation writes

$$2\sigma v(y) = \sigma(v(y-1) + v(y+1)) + f(v(y-1)) - f(v(y+1)),$$

which may be rewritten as

$$f(v(y)) + f(v(y+1)) + \sigma(v(y) - v(y+1)) = \text{ cst } = f(u^{l,r}), \tag{3.7}$$

where $f(u^r) = f(u^l)$. Let us apply (3.7) at $j+h$ and j and let make the difference of both equalities :

$$f(v(j+h+1)) - f(v(j+1)) + f(v(j+h)) - f(v(j)) = \sigma(v(j+h+1) - v(j+h) - v(j+1) + v(j)).$$

Let sum over $j \in \mathbb{Z}$; the right hand sides gives $\sigma([u] - [u])$, which is zero. Let now assume that a component $g := f_l$ is linear. Then we obtain

$$g\left(\sum_j (v(j+h+1) - v(j+1) + v(j+h) - v(j))\right) = 0,$$

that is $g(2V(h)) = 0$, or $g(V(h)) = 0$.

Let now assume that $f_1, \dots, f_{n-1}$ are independent linear forms. Then it follows from the previous calculation that $V(h)$ belongs to the kernel of $(f_1, \dots, f_{n-1})$, which is a one-dimensional subspace. Since the Rankine-Hugoniot condition $[f(u)] = 0$ implies $f_l([u]) = 0$ for $l < n$, one observes that $[u]$ belongs to the same kernel. Therefore, $V(h)$ and $[u]$ are colinear.

Since moreover the profile may be reparametrized (because $\eta = 0$) as $y \mapsto v \circ k(y)$, where k is any continuous function such that $k(y+1) = k(y) + 1$, one may choose k in such a way that the new profile satifies $V(h) = h[u]$. ∎

Chapter 4

Stability of viscous profiles

Here one adresses the question of asymptotic stability of a viscous shock profile as time goes to $+\infty$. Let

$$\partial_t u + \partial_x f(u) = \partial_x (B(u)\partial_x u) \tag{4.1}$$

be a parabolic system of conservation laws and let $u(x,t) := U(x-st)$ be a shock profile, with $U(\pm\infty) = u^{l,r}$. Up to a change of variables $(x,t) \mapsto (x-st,t)$, one always may assume that the shock is steady ($s=0$), although this has the effect to change the flux into $f(u) - su$.

From now on, we shall always assume that $s = 0$, so that U is a stationary solution :

$$B(U)U' = f(U) - f^*, \quad f^* := f(u^l) = f(u^r). \tag{4.2}$$

The general problem of this chapter is the following. Let consider the Cauchy problem for (4.1) with an initial data $U + \phi$, where ϕ is a disturbance, which may be small, smooth and/or decaying at infinity or whatever is convenient. Does the solution $u(t) := u(\cdot,t)$ approach U as $t \to +\infty$? We shall see that the answer is *no* in general, because the asymptotic beheviour is slightly more involved. Thus we shall reformulate the problem and then give a few answers.

4.1 The asymptotics as the time goes to $+\infty$

The analysis of this section is due to Tai-Ping Liu [8]. Hereabove, we do not consider the question of global-in-time existence of u. we just assume that such a solution exists, which in general will be smooth, because of regularizing properties of parabolic operators.

The first remark is that, whenever ϕ is integrable and decays at infinity, a formal calculation holds. The difference $u(t) - U$ remains integrable and

$$\frac{d}{dt}\int_{I\!R}(u(x,t) - U(x))dx = [B(u)\partial_x u - B(U)U' + f(U) - f(u)]_{-\infty}^{+\infty}.$$

The right-hand side vanishes since $\partial_x u$ and U' vanish at infinity and since U and u have the same limits. Thus the integral remains constant :

$$\int_{I\!R}(u(x,t) - U(t))dx = \int_{I\!R}\phi(x)dx. \tag{4.3}$$

Because of (4.3), $u(t)$ can converge in $L^1(I\!R)$ towards U only if the total mass of the initial disturbance is zero, which is unlikely.

One next may think that $u(t)$ might converge to a translated profile $U(\cdot + h)$ (still a stationary solution), but then

$$\int_{I\!R}(u(x,t) - U(x+h))dx = \int_{I\!R}(\phi(x) + U(x) - U(x+h))dx = \int_{I\!R}\phi(x)dx - h(u^r - u^l). \tag{4.4}$$

From this equality, we conclude that a necessary condition for the L^1-convergence towards a translated profile is that the mass of the disturbance be colinear to $[u]$, which is unlikely if $n \geq 2$. We shall see however in the next section that such a convergence always holds for scalar equations ($n = 1$).

When $n = 2$, we thus need somme object to compensate the mass of ϕ when it is not colinear to $[u]$. Actually, this difficulty already occurs when looking at the stability of constant states. Let a be a constant background state and let $a + \psi$ be the initial data, where $\psi \in L^1(I\!\!R)$ decays at infinity. Then again,

$$\int_{I\!\!R} (u(x,t) - a)dx = \int_{I\!\!R} \psi dx =: m_0.$$

The clue is that $u(t)$ tends uniformly to a as $t \to +\infty$, although it cannot converge in L^1 if $m_0 \neq 0$. This phenomenon already occurs for the heat equation $\partial_t u = \partial_x^2 u$, where $u(x,t) - a \sim t^{-1/2} z(x/\sqrt{t})$, where z is a Gaussian function.

4.1.1 Stability of constant states

When $u(x,0) = a + \psi(x)$, we thus consider an asymptotic expansion of the form

$$u(x,t) - a = \frac{1}{\sqrt{t}} \sum_{p=1}^{n} W_p \left(\frac{x - c_p t}{\sqrt{t}} \right) + \mathcal{O}\left(\frac{1}{t} \right).$$

The rôle of the travelling variables is to recall the finite speed propagation in the hyperbolic part $\partial_t u + \partial_x f(u)$. Besides, we have introduced n terms because (4.1) is a system of n conservation laws. One assumes that the c_p are pairwise distinct and that the W_p decay exponentially fast at infinity, so that the corresponding waves may be viewed as non-interacting patterns, located in the domains $D_p := \{|x - c_p t| = \mathcal{O}(\sqrt{t})\}$.

Let us focus on the domain D_p, where

$$u - a \sim \frac{1}{\sqrt{t}} W_p \left(\frac{x - c_p t}{\sqrt{t}} \right). \tag{4.5}$$

We introduce the travelling variable $y = x - c_p t$ and rewrite the system as

$$\partial_t u + \partial_y (f(u) - c_p u) = \partial_y (B(u) \partial_y u). \tag{4.6}$$

Using the asymptotic expansion, we first infer that

$$(df(a) - c_p) W_p' = 0,$$

so that c_p is an eigenvalue $\lambda_p(a)$ of $df(a)$, whereas $W_p(y) = w_p(y) r_p(a)$, where w_p is a scalar function.

Next, we multiply (4.6) by the eigen-form $l_p(a)$. Using again the asymptotic expansion and retaining the leading order terms (those in $t^{-3/2}$), we obtain the following equation for w_p :

$$\beta_p w_p'' = b_p w_p w_p' - \frac{1}{2}(w_p + \xi w_p'), \tag{4.7}$$

where ξ stands for the argument of w_p. Hereabove, $\beta_p := l_p(a) B(a) r_p(a)$, which we shall assume to be strictly positive, whereas $b_p = l_p D^2 f(r_p, r_p) = 2 d\lambda_p \cdot r_p$ describes the linearity (if $b_p = 0$) or the non-linearity (if $b_p \neq 0$) of the p-th characteristic field. One may integrate once the equation (4.7) as follow

$$2\beta_p w' = b_p w_p^2 - \xi w_p. \tag{4.8}$$

This ODE admits solutions which decay exponentially fast at infinity. Actually, given a real number m_p, it admits one and only one such solution satisfying the condition

$$\int_{\mathbb{R}} w_p(y)dy = m_p.$$

We shall denote by $w_p(y; m_p)$ this solution.

The important point is that the function $t^{-1/2}w_p((x-\lambda_p(a)t)/\sqrt{t}; m_p)r_p(a)$ has a constant mass $m_p r_p(a)$. Thus

$$\lim_{t\to+\infty}\int_{\mathbb{R}}(u(x,t)-a)dx = \sum_{p=1}^{n} m_p r_p(a),$$

which yields the necessary condition

$$\sum_{p=1}^{n} m_p r_p(a) = \int_{\mathbb{R}} \psi(x)dx, \tag{4.9}$$

since the integral of $u(t)-a$ is constant in time.

Here is the determination of the asymptotic behaviour in L^1, when the background state a is constant. The equality (4.9) determines the numbers $m_1, \dots, m_n$, since $(r_1(a), \dots, r_n(a))$ is a basis. Then each m_p determines the $w_p(\cdot; m_p)$, thus determines the "diffusion wave" W_p. Then (4.5) gives the answer. Let us remark that $u(t)$ is expected to converge to a in $L^q(\mathbb{R})$ for all $q > 1$, so that this analysis might look irrelevant or too much involved. We shall see that it is not, when considering the stability of a shock profile.

4.1.2 The case of a profile for a Lax shock

In the previous subsection, the initial mass were split into n diffusion waves $t^{-1/2}W_p((x - \lambda_p(a)t)/\sqrt{t})$, which travel at the pairwise distinct velocities $\lambda_p(a)$. We now introduce such waves in order to carry the mass which could not be balanced by a translation of the profile. The background state will be either $a = u^l$ if $\lambda_p(u^l) < 0$, or $a = u^r$ if $\lambda_p(u^r) > 0$, since these waves will escape from the left or from the right.

Assuming that our steady shock satisfies the Lax shock inequality

$$\lambda_q(u^r) < 0 < \lambda_q(u^l), \quad \lambda_{q-1}(u^l) < 0 < \lambda_{q+1}(u^r),$$

we thus have $q-1$ waves going to the left, associated to $p = 1, \dots, q-1$ and carrying masses $m_p r_p(u^l)$. Similarly, $n-q$ waves go to the right, associated to $p = q+1, \dots, n$ and carry masses $m_p r_p(u^r)$. Furthermore, we assume that, up to these waves, $u(t)$ converges towards a translated profile $U(\cdot + h)$. Then a straightforward calculation gives

$$\lim_{t\to+\infty}\int_{\mathbb{R}}(u(x,t)-U(x+h))dx = \sum_{1}^{q-1} m_p r_p(u^l) + \sum_{q+1}^{n} m_p r_p(u^r).$$

Since the right-hand side is actually constant, this implies the following identity :

$$\sum_{1}^{q-1} m_p r_p(u^l) + \sum_{q+1}^{n} m_p r_p(u^r) + h(u^r - u^l) = \int_{\mathbb{R}} \phi(x)dx. \tag{4.10}$$

As in the previous paragraph, (4.10) determines in a unique way the n real numbers $(m_1, \dots, m_{q-1}, h, m_{q+1}, \dots, m_n)$, provided the following generic condition holds

$$\det(r_1(u^l), \dots, r_{q-1}(u^l), u^r - u^l, r_{q+1}(u^r), \dots, r_n(u^r)) \neq 0. \tag{4.11}$$

Then each m_p determines a diffusion wave W_p, whereas h is responsible for a shift of the profile. The asymptotic behaviour is thus described by the following expansion, which is expected to hold in L^1

$$u(x,t)-U(x+h)\sim\sum_1^{q-1}\frac{1}{\sqrt{t}}w_p\left(\frac{x-\lambda_p(u^l)t}{\sqrt{t}};m_p\right)r_p(u^l)+\sum_{q+1}^{n}\frac{1}{\sqrt{t}}w_p\left(\frac{x-\lambda_p(u^r)t}{\sqrt{t}};m_p\right)r_p(u^r).$$

Again, we expect that $u(t)$ converges uniformly to $U(\cdot+h)$, but this analysis is usefull, since it gives rise to (4.10), which is needed for the determination of h.

4.2 The scalar case

The scalar case is much more favourable, because of the strong properties of the semi-group defined by (4.1). For the sake of simplicity, we now assume that $B(u)\equiv 1$. Then the Cauchy problem for (4.1), with a bounded measurable initial data u_0, admits one and only one bounded solution $u(x,t)$. The semi-group $(S(t))_{t\geq 0}$ is well-defined on $L^\infty(\mathbb{R})$ by the formula $S(t)u_0=u(t)$. Besides the classical properties $S(0)=\mathrm{id}_{L^\infty}$, $S(t+s)=S(t)\circ S(s)$, one has the following list.

SG1 (Smoothing) : let t be >0 and $a\in L^\infty$, then $S(t)a$ is as much differentiable as f.

SG2 (Maximum principle) : let $a\leq b$ almost everywhere, then $S(t)a\leq S(t)b$.

SG3 (Mass conservation) : let $b-a$ be integrable. Then $S(t)b-S(t)a$ is integrable and $\int_{\mathbb{R}}(S(t)b-S(t)a)dx=\int_{\mathbb{R}}(b-a)dx$.

SG4 (L^1-contraction) : moreover, $\|S(t)b-S(t)a\|_1\leq\|b-a\|_1$ (here $\|\cdot\|_q$ denotes the L^q-norm).

SG5 (equality case) : if a,b are of class $\mathcal{C}^2$, if $b-a\in L^1$ and if

$$\frac{d}{dt}\|S(t)b-S(t)a\|_1\Big|_{t=0}=0$$

then $b'=a'$ at every point where $b=a$.

SG6 (Matano) : Let $N(t)$ be the number of sign changes in the function $x\mapsto u(x,t)-a$ (a being some constant). Then $t\mapsto N(t)$ is a non-increasing function.

Thanks to the contraction property, we may extend in a unique way the semi-group to L^1+L^∞ with the property that it is continuous on each leaf $a+L^1$ (where $a\in L^\infty$), for the distance induced by L^1 : $d(b,c):=\|c-b\|_1$.

Let U be a stationary viscous profile between u^l and u^r, that is $U'=f(U)-f^*$. Then U is monotonous and the shock satisfies the Oleĭnik criterion : the graph of f is above (resp. below) f^* between u^l and u^r, if $u^l<u^r$ (resp. $u^r<u^l$). In particular, the shock is a Lax shock, in the wide sense : $f'(u^r)\leq 0\leq f'(u^l)$. Thus we do not expect any diffusion waves as $t\to+\infty$, but only a shift of the profile U, given by

$$h:=\frac{1}{u^r-u^l}\int_{\mathbb{R}}\phi(x)dx.$$

Up to a translation of the profile, we may assume that $\int_{\mathbb{R}}\phi(x)dx=0$, so that we expect the convergence of $u(t)$ towards U. The following result may be found in [3].

Theorem 4.2.1 *Let U be a steady solution of the scalar viscous conservation law $\partial_t u + \partial_x f(u) = \partial_x^2 u$, where $f \in \mathcal{C}^2(I\!R)$, and assume that U has limits $u^{l,r}$ at $\pm\infty$ (with $u^r \neq u^l$).*

Let ϕ be an L^1-function and $u(t) = S(t)(U + \phi)$ be the solution of the Cauchy problem associated to the initial data $U + \phi$. Then

$$\lim_{t \to +\infty} \|u(t) - U(\cdot + h)\|_1 = 0,$$

where h is defined by

$$h := \frac{1}{u^r - u^l} \int_{I\!R} \phi(x)dx.$$

The strength of this theorem is that it does not require any assumption on the initial data : neither smoothness, nor smallness. The counterpart is that no decay rate can be obtain in such a general setting. More powerfull results under stronger hypotheses were obtained by Ilin and Oleĭnik, by Sattinger, by Osher and Ralston and many other authors in the past few years.

Let us point out that, as quoted in the previous section, this theorem is partly the consequence of a result concerning the perturbation of constant states :

Theorem 4.2.2 *Let $a \in I\!R$ be a constant and let $\phi \in L^1$ be such that*

$$\int_{I\!R} \phi(x)dx = 0.$$

Then

$$\lim_{t \to +\infty} \|u(t) - a\|_1 = 0.$$

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